Brief Solution for Homework 3

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I. Hornstein (1993) JME

1)

The production function of intermediate goods firm is

$$x(j) = z\Big[k(j)^{\theta}h(j)^{1-\theta}\Big]^{\gamma} - \phi. \tag{1}$$

Figure 1 shows numerical examples of (a) and (b). The real lines show (a) and (b). For (a), we set $\gamma = 2$, and we set $\phi = .5$ for (b). We draw the graph by setting $(k_t(j), h_t(j)) = (\alpha, \alpha)$. The dotted lines are constant returns to scale examples such that they have the same values as those of (a) and (b) if $\alpha = 1$. The main distinction is that the shape of (b) is linear while that of (a) is convex.

2)

Here, we must make sure that the symmetrization is optimal. The recursive formulation of the social planner's problem is

$$\begin{aligned} V(K,z) = \max_{C,H,k(j),h(j),I} & \Big\{ \log C + \alpha \log(T-H) + \beta E\Big[V(K',z')\Big| z\Big]\Big\}, \\ \text{s.t. } & x(j) = z\big[k(j)^{\theta}h(j)^{1-\theta}\big]^{\gamma} - \phi, \\ & \log z' = \rho \log z + \varepsilon', \quad \varepsilon \sim N(0,\sigma^2), \\ & Y = \Big[\int_0^1 x(j)^{1/\mu}dj\Big]^{\mu}, \\ & C + I \leq Y, \\ & K' = K^{\lambda}I^{1-\lambda}, \\ & \int_0^1 h(j)dj \leq H, \\ & \int_0^1 k(j)dj \leq K. \end{aligned} \tag{2}$$

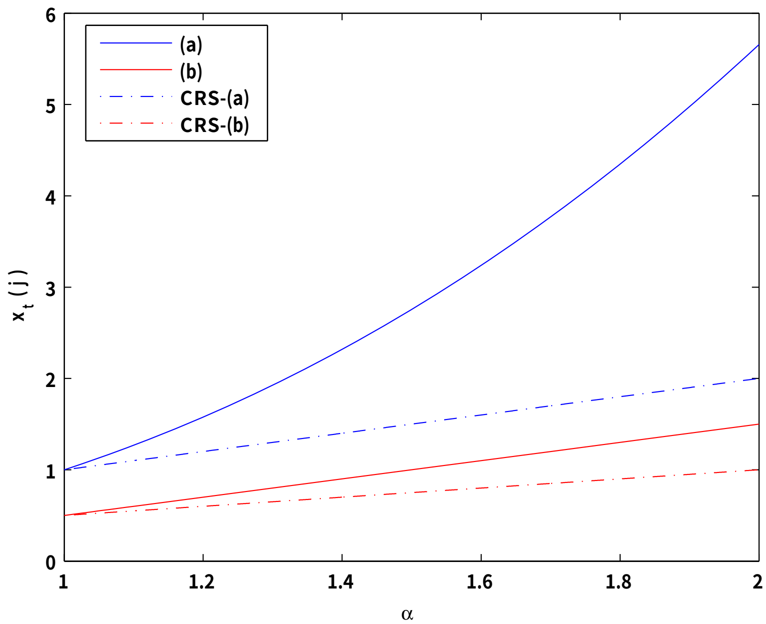

Figure 1: Two cases of Increasing Returns to Scale

Define the Lagrangian:

$$
\begin{aligned}
L = &\log\left(\left[\int_0^1 \left\{z\left[k(j)^\theta h(j)^{1-\theta}\right]^\gamma - \phi\right\}^{1/\mu} dj\right]^\mu - K'^{\frac{1}{1-\lambda}} K^{-\frac{\lambda}{1-\lambda}}\right) \\
&+ \alpha \log(T-H) + \beta E\left[V(K',z')\middle| z\right] - \Psi_h\left[\int_0^1 h(j)dj - H\right] \\
&- \Psi_k\left[\int_0^1 k(j)dj - K\right], \quad \big(\text{s.t. the equation (2)}\big)
\end{aligned}
$$

where Ψ_h and Ψ_k are Lagrange multipliers. Here, note that K is not a control variable while $k(j)$, $h(j)$, K' and H are[1].

[1] You can formulate the Lagrangian as

$$
\begin{aligned}
L = &\log\left(\left[\int_0^1 \left\{z\left[k(j)^\theta h(j)^{1-\theta}\right]^\gamma - \phi\right\}^{1/\mu} dj\right]^\mu - K'^{\frac{1}{1-\lambda}} K^{-\frac{\lambda}{1-\lambda}}\right) \\
&+ \alpha \log\left(T - \int_0^1 h(j)dj\right) + \beta E\left[V(K',z')\middle| z\right] - \Psi_k\left[\int_0^1 k(j)dj - K\right],
\end{aligned}
$$

but you *can not* formulate it as

$$
\begin{aligned}
L = &\log\left(\left[\int_0^1 \left\{z\left[k(j)^\theta h(j)^{1-\theta}\right]^\gamma - \phi\right\}^{1/\mu} dj\right]^\mu - K'^{\frac{1}{1-\lambda}} \left(\int_0^1 k(j)dj\right)^{-\frac{\lambda}{1-\lambda}}\right) \\
&+ \alpha \log(T-H) + \beta E\left[V(K',z')\middle| z\right] - \Psi_h\left[\int_0^1 h(j)dj - H\right],
\end{aligned}
$$

since K is not control variable.

The FONC's are

$$K': \quad \frac{1}{C}\frac{1}{1-\lambda}K'^{\frac{\lambda}{1-\lambda}}K^{-\frac{\lambda}{1-\lambda}} = \beta E\left[V_K(K',z')\Big| z\right], \tag{3}$$

$$H: \quad \frac{\alpha}{T-H} = \Psi_h, \tag{4}$$

$$h(j): \quad \frac{1}{C}\left[\int_0^1 x(j)^{1/\mu}dj\right]^{\mu-1} x(j)^{\frac{1-\mu}{\mu}} z\gamma(1-\theta)k(j)^{\theta\gamma}h(j)^{\gamma(1-\theta)-1} = \Psi_h, \tag{5}$$

$$k(j): \quad \frac{1}{C}\left[\int_0^1 x(j)^{1/\mu}dj\right]^{\mu-1} x(j)^{\frac{1-\mu}{\mu}} z\gamma\theta k(j)^{\theta\gamma-1}h(j)^{\gamma(1-\theta)} = \Psi_k. \tag{6}$$

By (5) and (6), we find that capital-labor ratios are same, ψ, across the intermediate goods firms:

$$\psi \equiv \frac{h(j)}{k(j)} = \frac{1-\theta}{\theta}\cdot\frac{\Psi_k}{\Psi_h}. \tag{7}$$

Substituting (7) into (1) gives

$$\begin{aligned} x(j) &= z[k(j)^\theta(\psi k(j))^{1-\theta}]^\gamma - \phi \\ &= z\psi^{(1-\theta)\gamma}k(j)^\gamma - \phi. \end{aligned}$$

Then we have

$$y = \left[\int_0^1 \left[z\psi^{(1-\theta)\gamma}k(j)^\gamma - \phi\right]^{\frac{1}{\mu}}dj\right]^\mu. \tag{8}$$

Consider the Case **a** in question 1, that is the case of $\phi = 0$. (8) becomes

$$const. \times \left[\int_0^1 k(j)^{\frac{\gamma}{\mu}}dj\right]^\mu.$$

Thus, if $\mu \geq \gamma$, the optimal policy is

$$k(j) = k(i).$$

On the other hand, consider the Case **b** in question 1, that is the case of $\gamma = 1$. (8) becomes

$$const. \times \left\{\int_0^1 [k(j) - const.]^{\frac{1}{\mu}}dj\right\}^\mu.$$

Thus, the optimal policy is

$$k(j) = k(i).$$

Finally, we find that $k(j)$ and hence $h(j)$ are same across intermediate goods firms, and therefore $x(j)$ are also same across firms. Note that $k(j)$ may be different from $k(i)$ if $\mu < \gamma$ ($h(j)$ and $x(j)$, too.) in case **a**, so that this condition is crucial.

Therefore, the recursive formulation of the social planner's problem under a symmetric assumption is

$$V(K,z) = \max_{C,H,I}\left\{ \log C + \alpha \log(T-H) + \beta E\left[V(K',z')\middle| z\right]\right\}, \tag{9}$$

$$\text{s.t. } X = z\left[K^{\theta}H^{1-\theta}\right]^{\gamma} - \phi, \tag{10}$$

$$\log z' = \rho \log z + \varepsilon', \quad \varepsilon \sim N(0,\sigma^2), \tag{11}$$

$$Y = X, \tag{12}$$

$$C + I \leq Y, \tag{13}$$

$$K' = K^{\lambda} I^{1-\lambda}. \tag{14}$$

3)

The cost minimization problem for final goods firms is

$$\min_{x(\cdot)} \int_0^1 v(j)x(j)dj,$$

$$\text{s.t. } \left[\int_0^1 x(j)^{1/\mu}dj\right]^{\mu} \geq y.$$

The F.O.N.C is

$$x(j): \quad v(j) - \lambda\left[\int_0^1 x(j)^{1/\mu}dj\right]^{\mu-1} x(j)^{\frac{1-\mu}{\mu}} = 0, \tag{15}$$

$$\lambda: \quad \left[\int_0^1 x(j)^{1/\mu}dj\right]^{\mu} = y, \tag{16}$$

where λ is the Lagrange multiplier. By (15), we obtain

$$\frac{x(j)}{x(i)}: \quad \frac{v(j)}{v(i)} = \left[\frac{x(j)}{x(i)}\right]^{\frac{1-\mu}{\mu}},$$

$$\iff \quad x(j) = \left[\frac{v(j)}{v(i)}\right]^{\frac{\mu}{1-\mu}} x(i). \tag{17}$$

By (16) and (17),

$$\left[\int_0^1 \left\{ \left[\frac{v(j)}{v(i)}\right]^{\frac{\mu}{1-\mu}} x(i)\right\}^{1/\mu} dj\right]^{\mu} = y,$$

$$\iff \quad x(i) = \left[\int_0^1 v(j)^{\frac{1}{1-\mu}}dj\right]^{-\mu} v(i)^{\frac{\mu}{1-\mu}} y. \tag{18}$$

This can be rewritten as

$$D_j^F(v,y) = \left[\int_0^1 v(s)^{\frac{1}{1-\mu}} ds\right]^{-\mu} v(j)^{\frac{\mu}{1-\mu}} y. \tag{19}$$

4)

The cost minimization problem for a typical intermediate goods firm j is

$$\min_{k(j),h(j)} \left\{ wh(j) + rk(j) \;\middle|\; k(j)^\theta h(j)^{1-\theta} = \left(\frac{x+\phi}{z}\right)^{1/\gamma} \right\},$$

$$\iff \quad \min_{k(j)} \quad wk(j)^{\frac{\theta}{\theta-1}} \left(\frac{x+\phi}{z}\right)^{\frac{1}{\gamma(1-\theta)}} + rk(j).$$

You can solve this problem straightforwardly. Then, we obtain

$$k(j) = \left[\frac{r}{w}\cdot\frac{1-\theta}{\theta}\right]^{\theta-1} \left[\frac{x+\phi}{z}\right]^{1/\gamma}, \tag{20}$$

$$h(j) = \left[\frac{r}{w}\cdot\frac{1-\theta}{\theta}\right]^{\theta} \left[\frac{x+\phi}{z}\right]^{1/\gamma}. \tag{21}$$

5)

The profit maximization problem for an intermediate goods firm j is

$$\max_{v(j)} v(j) D_j^F(v,y) - C_j[r,w,D_j^F(v,y),z], \tag{22}$$

where $D_j^F(v,y)$ is given by (19) and $C_j[r,w,D_j^F(v,y),z]$ is the cost function which will be given in the next question.

6)

The cost function for a intermediate goods firm is

$$C_j(r,w,x,z) = wh(j) + rk(j). \tag{23}$$

Here $k(j)$ and $h(j)$ satisfy (20) and (21) respectively.

So, the explicit form of the cost function is given by

$$C_j(r,w,x,z) = \left[\frac{r}{w}\cdot\frac{1-\theta}{\theta}\right]^{\theta} w \frac{1}{1-\theta} \left[\frac{x+\phi}{z}\right]^{1/\gamma}.$$

Thus, we obtain

$$\begin{aligned} MC_j \equiv& \frac{\partial C_j(r,w,x,z)}{\partial x} \\ =& \frac{1}{\gamma}\theta^{-\theta}(1-\theta)^{\theta-1} r^\theta w^{1-\theta} z^{-1/\gamma} (x+\phi)^{\frac{1}{\gamma}-1}, \end{aligned} \tag{24}$$

by a straightforward calculation.

7)

CAUTION: Just solving (22) mathematically might not give us the expression (1.11). We need an economic intuition to derive it.

Rewrite (22) as

$$\max_{x(j)} v_j[x(j), y]x(j) - C_j[r, w, x(j), z], \tag{25}$$

where $v_j[x(j), y]$ is the price function. If we have it, it is easy to prove that this is identical to (22). Considering (18), the function is given by

$$v_j[x(j), y] = x(j)^{\frac{1-\mu}{\mu}} \left[\int_0^1 v(j)^{\frac{1}{1-\mu}} dj \right]^{1-\mu} y^{\frac{\mu-1}{\mu}}.$$

Note that the second term of RHS is constant since the price of the final good is given. And this fact is the crucial point to derive the required expression.

The FONC for (25) is

$$v_j(x(j), y) + x(j) \frac{\partial v_j(x(j), y)}{\partial x(j)} - MC_j = 0.$$

Of course, this is well-known relation in basic microeconomics; the marginal revenue (i.e., sum of the first and second term) is equal to the marginal cost. Therefore, we obtain[2]

$$\begin{aligned} & v_j(x(j), y) + \frac{1-\mu}{\mu} v_j(x(j), y) = MC_j \\ \Leftrightarrow\ & \frac{v(j)}{MC_j} = \mu. \end{aligned}$$

Note that the markup is strictly larger than one in this economy.

8a)

Setting $p = v = 1$, (24) becomes

$$1 = \frac{\mu}{\gamma} \left[\frac{r}{w} \cdot \frac{1-\theta}{\theta} \right]^{\theta} \frac{w}{1-\theta} \left[\left(k^{\theta} h^{1-\theta} \right)^{\gamma} \right]^{1/\gamma - 1} \frac{1}{z}. \tag{26}$$

By (20), (21), (26) and (1), we have

$$w(k, h, z) = (1-\theta) \frac{\gamma}{\mu} z \left(k^{\theta} h^{1-\theta} \right)^{\gamma} / h, \tag{27}$$

$$r(k, h, z) = \theta \frac{\gamma}{\mu} z \left(k^{\theta} h^{1-\theta} \right)^{\gamma} / k. \tag{28}$$

[2]Here, we utilize the similar economic intuition that the change of input of one intermediate good has no effect the amount of aggregate output.

By (27) and (28),

$$\Pi(k,h,z) \equiv \left[\left(k^{\theta}h^{1-\theta}\right)^{\gamma} - \phi\right] - rk - wh,$$

$$= \left[1 - \frac{\gamma}{\mu}\right] z\left(k^{\theta}h^{1-\theta}\right)^{\gamma} - \phi. \tag{29}$$

8b)

The recursive household problem is

$$\begin{aligned} v(K,k,z) = \max_{c,h,i} & \left\{ \log c + \alpha \log(T-h) + \beta E\left[v(K',k',z') \middle| z \right] \right\}, \\ \text{s.t. } & \log z' = \rho \log z + \varepsilon', \quad \varepsilon' \sim N(0,\sigma^2), \\ & c + i = r(K,H,z)k + w(K,H,z)h + \pi, \\ & k' = k^{\lambda} i^{1-\lambda}, \\ & K' = \boldsymbol{G}(K), \\ & H = \boldsymbol{H}(K,z), \\ & \pi = \boldsymbol{\pi}(K,H,z), \end{aligned}$$

where π denotes return from firms.

9)

Definition: *A symmetric recursive (monopolicstic competitive) rational expectation equilibrium* is a collection of
(i) household's policy functions: $c(K,k,z)$, $h^s(K,k,z)$, $i(K,k,z)$, and $k^s(K,k,z)$,
(ii) household's value function: $v(K,k,z)$,
(iii) final goods firms' decision rule: $x(v)$
(iv) intermediate goods firms' decision rules: $v(K,z)$, $k^d(K,z)$ and $h^d(K,z)$,
(v) price system: $p(K,z)$, $v(K,z)$, $r(K,H,z)$ and $w(K,H,z)$,
such that

- (Households Optimization) Given $\{r(K,H,z), w(K,H,z)\}$, households policy functions: $c(K,k,z)$, $h(K,k,z)$, $i(K,k,z)$, and $k'(K,k,z)$, solve the recursive households problem in the question (8), and they achieve $v(K,k,z)$.

- (Final Goods Firm Optimization) Given $\{p(K,z), v(K,z)\}$, $x(v)$ solves the final goods firms' problem:

$$\max_{x} \quad py - vx,$$
$$\text{s.t. } y = x.$$

- (Intermediate Goods Firm Optimization) Given $\{r(K,H,z), w(K,H,z)\}$, intermediate goods firms' decision rule: $v(r,w,z,x)$, $k^d(r,w,z,x)$ and $h^d(r,w,z,x)$, solve their profit optimization problem in the question (5).

- (Market Clear) Households policy funciton satisify

$$\begin{aligned} C &= c(K,k,z), \\ H &= h^s(K,k,z), \\ I &= i(K,k,z). \end{aligned}$$

  Firms decision rules satisfy

$$\begin{aligned} K &= k^d(r,w,z,x), \\ H &= h^d(r,w,z,x), \\ Y &= x(v). \end{aligned}$$

  Finally, resource constraint is satisfied:

$$C + I = Y.$$

- (Rational Expectation) Agents' conjecture of the evolution of aggregate variables is correct:

$$k^s(K,K,z) = \boldsymbol{G}(K,z).$$

10)

[Social Planner's Solution] The FONC's for the recursive social planner's problem under symmetric assumption, (9)-(14), are

$$K': \qquad \frac{1}{C}\frac{1}{1-\lambda}K'^{\frac{\lambda}{1-\lambda}}K^{-\frac{\lambda}{1-\lambda}} = \beta E\left[V_K(K',z')\middle| z\right], \tag{30}$$

$$H: \qquad \frac{1}{C}z\gamma(1-\theta)K^{\theta\gamma}H^{\gamma(1-\theta)-1} = \frac{\alpha}{T-H}. \tag{31}$$

The Benveniste-Scheinkman formula is

$$V_K(K,z) = \frac{1}{C}\Big(\gamma\theta z K^{\theta\gamma-1}H^{\gamma(1-\theta)} + \frac{\lambda}{1-\lambda}K'^{\frac{1}{1-\lambda}}K^{-\frac{1}{1-\lambda}}\Big).$$

Then, the system for the optimal allocation is a collection of equations:

$$\begin{aligned} \text{(Euler Eq.)} \quad & \frac{1}{C}\frac{1}{1-\lambda}K'^{\frac{\lambda}{1-\lambda}}K^{-\frac{\lambda}{1-\lambda}} \\ &= \beta E\left[\frac{1}{C'}\Big(\gamma\theta z K'^{\theta\gamma-1}H'^{\gamma(1-\theta)} + \frac{\lambda}{1-\lambda}K''^{\frac{1}{1-\lambda}}K'^{-\frac{1}{1-\lambda}}\Big)\middle| z\right], \end{aligned} \tag{32}$$

$$\text{(Intra-temporal)} \quad \frac{1}{C}z\gamma(1-\theta)K^{\theta\gamma}H^{\gamma(1-\theta)-1} = \frac{\alpha}{T-H}, \tag{33}$$

and the resource constraint.

[Decentralized Economy] The FONC's for recursive household problem are

$$\begin{aligned} k' : &\quad \frac{1}{c}\frac{1}{1-\lambda}k'^{\frac{\lambda}{1-\lambda}}k^{-\frac{\lambda}{1-\lambda}} = \beta E\left[v_k(K',k',z',) \middle| K,z \right], \\ h : &\quad \frac{1}{c}w = \frac{\alpha}{T-h}. \end{aligned}$$

The Benveniste-Scheinkman formula is

$$v_k(K,k,z) = \frac{1}{c}\left[r + \frac{\lambda}{1-\lambda}k'^{\frac{1}{1-\lambda}}k^{-\frac{1}{1-\lambda}} \right].$$

The reduced form of MP conditions are (27) and (28).
Then, the equlibrium system is

$$\begin{aligned} \text{(Euler Eq.)} \quad & \frac{1}{c}\frac{1}{1-\lambda}k'^{\frac{\lambda}{1-\lambda}}k^{-\frac{\lambda}{1-\lambda}} \\ &= \beta E\left[\frac{1}{c'}\left(\gamma\theta\frac{1}{\mu}zk'^{\theta\gamma-1}h'^{\gamma(1-\theta)} + \frac{\lambda}{1-\lambda}k''^{\frac{1}{1-\lambda}}k'^{-\frac{1}{1-\lambda}} \right) \middle| K,z \right], \end{aligned} \tag{34}$$

$$\text{(Intra-temporal)} \quad \frac{1}{c}\frac{1}{\mu}z\gamma(1-\theta)k^{\theta\gamma}h^{\gamma(1-\theta)-1} = \frac{\alpha}{T-h}, \tag{35}$$

and the resource constraint.

Comparing (32) and (33) with (34) and (35), we find that the decentralized economy is the same as that of social planner's solution if $\mu = 1$.

11)

Consider the following fiscal policy;

- negative labor income tax: $1-\mu$
- negative capital income tax: $1-\mu$
- lump-sum tax: T (Balanced Budget: $(1-\mu)Hw + (1-\mu)rK = T$)

Under this policy, the budget constraint for household is

$$c + i = \mu r(K,H,z)k + \mu w(K,H,z)h + \pi - T.$$

Then, the FONC's and Benveniste-Scheinkman formula become

$$\begin{aligned} k' : &\quad \frac{1}{c}\frac{1}{1-\lambda}k'^{\frac{\lambda}{1-\lambda}}k^{-\frac{\lambda}{1-\lambda}} = \beta E\left[v_k(K',k',z',) \middle| K,z \right], \\ h : &\quad \frac{1}{c}\mu w = \frac{\alpha}{T-h}, \\ \text{(BS)} : &\quad v_k(K,k,z) = \frac{1}{c}\left[\mu r + \frac{\lambda}{1-\lambda}k'^{\frac{1}{1-\lambda}}k^{-\frac{1}{1-\lambda}} \right]. \end{aligned}$$

Using MP conditions, we obtain

$$\text{(Euler Eq.)} \quad \frac{1}{c}\frac{1}{1-\lambda}k'^{\frac{\lambda}{1-\lambda}}k^{-\frac{\lambda}{1-\lambda}} = \beta E\left[\frac{1}{c'}\gamma\theta z k'^{\theta\gamma-1}h'^{\gamma(1-\theta)} + \frac{\lambda}{1-\lambda}k''^{\frac{1}{1-\lambda}}k'^{-\frac{1}{1-\lambda}}\middle| K, z\right],$$

$$\text{(Intra-temporal)} \quad \frac{1}{c}z\gamma(1-\theta)k^{\theta\gamma}h^{\gamma(1-\theta)-1} = \frac{\alpha}{T-h}.$$

These are the same as the social planner's solution!

[Another Solution] You can also achieve the optimal allocation under an alternative policy;

- subsidy of labor cost to intermediate goods firm: $1-\frac{1}{\mu}$
- subsidy of capital cost to intermediate goods firm: $1-\frac{1}{\mu}$
- lump-sum tax to households: T
  (Balanced Budget: $T = (1-\frac{1}{\mu})(rK + wH)$)

Under this policy, we can derive analogues of (27) and (28), i.e.,

$$w(k,h,z) = (1-\theta)\gamma z\left(k^{\theta}h^{1-\theta}\right)^{\gamma}/h,$$
$$r(k,h,z) = \theta\gamma z\left(k^{\theta}h^{1-\theta}\right)^{\gamma}/k.$$

Finally, we can also obtain the same equations as (32) and (33).

II. Rossi-Hansberg + Wright (2006) mimeo

For simplicity, as announced, assume the identical fixed cost ϕ across industry.

1)

That is because the technology is diminishing returns to scale[3].

2)

The human capital for next period H' cannot be the control, since the shock to it, A', realizes in the next period. The recursive formation of the social planner's problem is as follows:

$$V(N, \mathbf{H}, \mathbf{A}) = \max_{\{N_j, X_j, \mu_j\}_{j=1}^J} \left\{ N \log \frac{C}{N} + \beta E\Big[V(N', \mathbf{H}', \mathbf{A}')|\mathbf{A}\Big] \right\}$$

$$\begin{aligned}
\text{s.t.} \quad & C = B \prod_{j=1}^{J} (Y_j - X_j)^{\theta}, \\
& Y_j = (H_j^{\eta} N_j^{1-\eta})^{\gamma} \mu_j^{1-\gamma} - \phi \mu_j, \\
& N = \sum_{j=1}^{J} N_j, \\
& N' = (1 + g_N) N, \\
& H_j' = A_j' H_j^{w} X_j^{1-w}, \\
& \log A_j' = \rho \log A_j + \epsilon', \\
& \epsilon' \sim N(0, \sigma^2), \quad \text{for all } j = 1, \cdots, J,
\end{aligned}$$

where $\mathbf{H} = (H_1, \cdots, H_J)$ and $\mathbf{A} = (A_1, \cdots, A_J)$.

3)

By substituting some constraints into the objective function appropriately, we can define the Lagrangian with a multiplier λ as follows:

$$L \equiv N \log \frac{B \prod_{j=1}^{J} \left\{ \Big[(H_j^{\eta} N_j^{1-\eta})^{\gamma} \mu_j^{1-\gamma} - \phi \mu_j \Big] - X_j \right\}^{\theta}}{N}$$

$$\beta E\Big[V(N', \mathbf{H}', \mathbf{A}')|\mathbf{A}\Big] + \lambda(N - \sum_{j=1}^{J} N_j),$$

[3]The assumption that all establishments experience the same shock of human capital is irrevant to this question.

where $N' = (1+g_N)N$, $H'_j = A'_j H_j^w X_j^{1-w}$, and $A'_j = A_j^\rho e^{\epsilon'}$.

Be careful!! Changing N_j affects the current utility through the consumption C, but does not through the population N. This is because N_j determines only the distribution of labor supply among industry but the aggregate labor supply is predetermined (N is a state).

The FONC's for the social planner's problem are

$$w.r.t. \quad N_j \quad : \qquad \gamma\theta(1-\eta)N\left[H_j^{\eta(\gamma-1)}N_j^{\gamma(1-\eta)-1}\mu_j^{1-\gamma}\right](Y_j - X_j)^{-1} = \lambda, \tag{36}$$

$$w.r.t. \quad \mu_j \quad : \qquad \theta N\left[(1-\gamma)\Big(\frac{H_j^\eta N_j^{1-\eta}}{\mu_j}\Big)^\gamma - \phi\right](Y_j - X_j)^{-1} = 0,$$

$$\Leftrightarrow \qquad (1-\gamma)\left(\frac{H_j^\eta N_j^{1-\eta}}{\mu_j}\right)^\gamma = \phi \tag{37}$$

$(\because Y_j \neq X_j,$ since $Y_j = X_j$ gives $C = 0$ and is obviously suboptimal.),

$$w.r.t. \quad X_j \quad : \qquad \frac{N\theta}{Y_j - X_j} = \beta E\left[V_{H_j}(N', \mathbf{H}', \mathbf{A}')(1-w)A'_j H_j^w X_j^{-w} | \mathbf{A}\right]. \tag{38}$$

4)

Key is whether the good with subscript i is distinctive from that with j. From this point of view, the number of human capitals traded is J, that of labor is 1, and that of intermediate goods is J. We cannot distinguish N_i from N_j. The prices of traded goods are r_j for H_j, ω for N, and p_j for Y_j respectively. By the way, you may think the consumption good C is traded in this economy, but obviously it is redundant.

5)

The establishment's problem is

$$\max_{h_j^d, n_j^d} p_j[(h_j^{d\,\eta} n_j^{d\,1-\eta})^\gamma - \phi] - r_j h_j^d - \omega n_j^d. \tag{39}$$

Here, we put superscript d on h and n in order to discriminate between supply and demand, which is needed in subsequent questions. The FOC's are

$$w.r.t. \quad h_j^d \quad : \qquad \frac{r_j}{p_j} = \eta\gamma h_j^{d\,\eta\gamma-1} n_j^{d\,(1-\eta)\gamma}, \tag{40}$$

$$w.r.t. \quad n_j^d \quad : \qquad \frac{\omega}{p_j} = (1-\eta)\gamma h_j^{d\,\eta\gamma} n_j^{d\,(1-\eta)\gamma-1}. \tag{41}$$

6)

Establishments enter into each industry unless they obtain zero profit. Thus (39) equals to zero at optimal. Substituting (40) and (41) into this non-profit condition, we obtain

$$\mu_j = H_j^{d\eta} N_j^{d1-\eta} \left(\frac{1-\gamma}{\phi}\right)^{\frac{1}{\gamma}}. \tag{42}$$

Here we utilize $H_j^d = \mu_j h_j^d$ and $N_j^d = \mu_j n_j^d$.

7)

The household's budget constraint is[4]

$$\sum_{j=1}^{J} p_j y_j^d \leq \sum_{j=1}^{J} r_j h_j^s + \omega N + \Pi, \tag{43}$$

where y_j^d denotes the demand of good j, h_j^s denotes the supply of human capital to industry j, and Π is profit by private ownership, which is zero in equilibrium.

8)

Assume $\Pi = 0$. The recursive representative household's problem is

$$\begin{aligned}
V(N, \mathbf{H}, \mathbf{A}, \mathbf{h^s}) = & \max_{\{y_j^d, x_j\}_{j=1}^J} \left\{ N \log \frac{C}{N} + \beta E\Big[V(N', \mathbf{H}', \mathbf{A}', \mathbf{h^s}')|\mathbf{A}\Big] \right\} \\
\text{s.t.} \quad & C = B \prod_{j=1}^{J} (y_j^d - x_j)^\theta, \\
& \sum_{j=1}^{J} p_j y_j^d \leq \sum_{j=1}^{J} r_j h_j^s + \omega N, \\
& N' = (1 + g_N) N, \\
& h_j^{s\prime} = A_j' h_j^{sw} x_j^{s1-w}, \\
& H_j' = A_j' H_j^w X_j^{1-w}, \\
& \log A_j' = \rho \log A_j + \epsilon', \\
& \epsilon' \sim N(0, \sigma^2), \\
& X_j = X_j(N, \mathbf{H}, \mathbf{A}) \quad \text{for all } j = 1, \cdots, J,
\end{aligned}$$

where $\mathbf{h^s} = (h_1^s, \cdots, h_J^s)$. $h_j^{s\prime}$ cannot be a control because the household cannot choose it directly.

[4] Do not confuse the household's budget constraint with any constraints of household's problem. Neither C nor X appears.

By substituting the first constraint into the objective function, we can define the Lagrangian with a multiplier ξ as follows:

$$L \equiv N \log \frac{B \prod_{j=1}^{J} (y_j^d - x_j)^\theta}{N} + E\Big[V(N', \mathbf{H}', \mathbf{A}', \mathbf{h^s}')|\mathbf{A}\Big] + \xi(\sum_{j=1}^{J} r_j h_j^s + \omega N - \sum_{j=1}^{J} p_j y_j^d),$$

where $N' = (1+g_N)N$, $h_j^{s\prime} = A_j' h_j^{s\,w} x_j^{s\,1-w}$, $H_j' = A_j' H_j^w X_j^{1-w}$, and $A_j' = A_j^\rho e^{\epsilon'}$.

The FONC's for the household's problem are

$$w.r.t. \quad y_j^d \quad : \qquad \frac{N\theta}{y_j^d - x_j} = \xi p_j, \tag{44}$$

$$w.r.t. \quad x_j \quad : \qquad \frac{N\theta}{y_j^d - x_j} = \beta E\left[V_{h_j^s}(N', \mathbf{H}', \mathbf{A}')(1-w) A_j' h_j^{s\,w} x_j^{-w} |\mathbf{A}\right]. \tag{45}$$

9)

For convenience, define the whole state vector $\Omega \equiv \{N, \mathbf{H}, \mathbf{A}\}$.

Definition: A *recursive competitive equilibrium* is a collection of
(i) household's policy functions: $x_j(\Omega, \mathbf{h^s})$, and $y_j^d(\Omega, \mathbf{h^s})$,
(ii) household's value function: $V(\Omega, \mathbf{h^s})$,
(iii) decision rules for each establishment in industry j : $h_j^d(\Omega)$, and $n_j^d(\Omega)$,
(iv) price system: $p_j(\Omega)$, $r_j(\Omega)$, and $\omega(\Omega)$,
for $j = 1, \cdots, J$, such that

- (Households Optimization) Given $r_j(\Omega)$,$\omega(\Omega)$, households policy functions solve the recursive households problem in the question (8), and they achieve $V(\Omega)$,

- (Establishments Optimization) Given $p_j(\Omega)$, $r_j(\Omega)$, $\omega(\Omega)$ establishment's decision rule solves its profit maximization problem in the question (5),

- (Market Clear)

$$y_j^d = y_j^s \equiv (h_j^{d\,\eta} n_j^{d\,1-\eta})^\gamma - \phi,$$
$$H_j^d = h_j^s,$$
$$\sum_{j=1}^{J} N_j^d = N.$$

- (Rational Expectation) Agents' conjecture is correct:

$$x_j(\Omega, \mathbf{H}) = X_j(\Omega).$$

In equilibrium, (45) and (42) imply (38) and (37) respectively. Moreover, substituting (41) into (44) gives (36) by setting $\lambda = \xi\omega$. Thus, the FOC's for the household's problem and an equilibrium necessary condition are identical with the the FOC's for the social planner's problem in equilibrium, and hence the competitive equilibrium is pareto optimal.

10) - 12)

The remaining questions are too difficult to prove our conjecture. Therefore, we show here the similation results using dynare instead of the mathematical proof.

We simply set $J = 3, N = 1, \beta = 0.96, \gamma = 0.5, \theta = 1.2, \eta = 0.36, \rho = 0.9, w = 2/3, \phi = 0.5, g_N = 0$. The SS values with technology 1 and 1.05 are as follows:

Simulation Result

	w/ $A_j = 1$	w/ $A_j = 1.05$
H_j	0.1686	0.2103
N_j	0.3333	0.3258
X_j	0.1686	0.1817
Y_j	0.0589	0.0710
μ_j	0.2608	0.2793

The employment decreases and the number of firms that operate increase. This is intuitive as a result of positive technology shock.

This result, however, might vary according to the parameter values. In these questions, even if your results are inconsistent to the result here, you can obtain full scores as long as you explain your conjecture logically.